

## A REMARK ON A SUM INVOLVING THE PRIME COUNTING FUNCTION

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We introduce explicit bounds for the sum  $\sum_{2 \leq n \leq x} 1/\pi(n)$ , where  $\pi(n)$  is the number of primes that are not greater than  $n$ .

### 1. INTRODUCTION AND MOTIVATION

Let  $\pi(x)$  be the number of primes not exceeding  $x$ . In 1980 JEAN-MARIE DE KONINCK and ALEKSANDAR IVIĆ [3] showed

$$\sum_{2 \leq n \leq x} \frac{1}{\pi(n)} = \frac{1}{2} \log^2 x + O(\log x),$$

as a consequence of the Prime Number Theorem.

In 2000, L. PANAITOPOL [5] proved that

$$\frac{1}{\pi(x)} = \frac{1}{x} \left( \log x - 1 - \sum_{i=1}^m \frac{k_i}{\log^i x} + O\left(\frac{1}{\log^{m+1} x}\right) \right),$$

where  $m \geq 1$  and  $\sum_{i=0}^{n-1} i! k_{n-i} = n.n!$ , and using this with  $m = 2$ , improved the above result of KONINCK and IVIĆ to

$$\sum_{2 \leq n \leq x} \frac{1}{\pi(n)} = \frac{1}{2} \log^2 x - \log x - \log \log x + O(1).$$

Two years later, IVIĆ [2] obtained

$$\sum_{2 \leq n \leq x} \frac{1}{\pi(n)} = \frac{1}{2} \log^2 x - \log x - \log \log x + C + \sum_{i=2}^m \frac{k_i}{(i-1) \log^{i-1} x} + O\left(\frac{1}{\log^m x}\right),$$

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where  $C$  is an absolute constant.

As we see, all above approximations of the sum  $\sum_{2 \leq n \leq x} \frac{1}{\pi(n)}$  have  $O$ -terms with effective constants, but there is no known computed one. In this note, following some careful computations, we find a lower and an upper bound for this sum. Our main result is:

**Theorem.** *For  $x \geq 2$  we have*

$$-1.51 \log \log x + 0.8994 < \sum_{2 \leq n \leq x} \frac{1}{\pi(n)} - \left( \frac{1}{2} \log^2 x - \log x \right) < -0.79 \log \log x + 6.4888.$$

REMARK

1. The factors of  $\log \log$  -terms and constants that appear in the statement of this theorem are not optimal, and could be further improved. But, these are best with our methods and computational tools.

2. It would be of an interest to determine the value of  $C$  in Ivić's expansion of the summation under study, which requires studying the sequence  $\{C_N\}_{N \geq 2}$  defined by

$$C_N = \sum_{n=2}^N \frac{1}{\pi(n)} - \frac{1}{2} \log^2 N + \log N + \log \log N.$$

Ivić's result, guarantees that  $C_N \rightarrow C$ , and computations show that  $C \approx 6.9$ .

## 2. PROOF OF THE THEOREM

First we need the EULER-MACLAURIN summation formula to get some lemmas. Let  $B_n(t)$  denotes BERNOULLI polynomials and let  $B_n = B_n(0)$  be BERNOULLI numbers; for example  $B_2(t) = t^2 - t + 1/6$  and  $B_2 = 1/6$ . By  $\{t\}$  we denote the fractional part of  $t$ . Considering the inequality  $|B_{2m}(\{t\})| \leq B_{2m}$ , and using the EULER-MACLAURIN summation formula (see for example ODLYZKO [4]) with  $m = 1$ , we get the following lemmas.

**Lemma 1.** *For  $N \geq 2$  we have*

$$\Sigma_{-1}(N) := \sum_{n=2}^N \frac{1}{n \log n} = \log \log N + C_{-1} + R_{-1},$$

where  $C_{-1} = 0.794678645452\dots$ , and  $|R_{-1}| \leq (3N \log N + \log N + 1)/(6N^2 \log^2 N)$ .

**Lemma 2.** *For  $N \geq 1$  we have*

$$\Sigma_0(N) := \sum_{n=1}^N \frac{1}{n} = \log N + \gamma + R_0,$$

where  $\gamma = 0.577215664901\dots$ , and  $|R_0| \leq (3N + 1)/(6N^2)$ .

**Lemma 3.** For  $N \geq 1$  we have

$$\Sigma_1(N) := \sum_{n=1}^N \frac{\log n}{n} = \frac{1}{2} \log^2 N + C_1 + R_1,$$

where  $C_1 = -0.072815845483\dots$ , and  $|R_1| \leq (6N \log N + 13 \log N + 25)/(12N^2)$ .

Let  $\mathfrak{S}(y, x) = \sum_{y < n \leq x} \frac{1}{\pi(n)}$  and  $\mathfrak{S}(x) = \mathfrak{S}(1, x)$ . To find an upper bound, we consider Theorem 2.1 of HASSANI [1], which asserts that for every  $0 < \varepsilon \leq \frac{21}{20}$ , the inequality

$$\frac{x}{\log x - 1 - \frac{\frac{4}{5} - \varepsilon}{\log x}} \leq \pi(x),$$

holds for all  $x$  such that

$$x \geq \max \left\{ 32299, \left\lceil e^{\frac{13 - 5\varepsilon + \sqrt{169 + 14\varepsilon - 155\varepsilon^2}}{10\varepsilon}} \right\rceil \right\} := \mathfrak{M},$$

say. We write  $\mathfrak{S}(x) = \mathfrak{S}(\mathfrak{M} - 1) + \mathfrak{S}(\mathfrak{M} - 1, x)$ . The value of  $\mathfrak{S}(\mathfrak{M} - 1)$  is absolute and one can find it after determining exact value of  $\varepsilon$ . We write

$$\begin{aligned} \mathfrak{S}(\mathfrak{M} - 1, x) &\leq \sum_{\mathfrak{M}-1 < n \leq x} \frac{\log n - 1 - \frac{\frac{4}{5} - \varepsilon}{\log n}}{n} \\ &= \sum_{\mathfrak{M}-1 < n \leq x} \frac{\log n}{n} - \sum_{\mathfrak{M}-1 < n \leq x} \frac{1}{n} - \left(\frac{4}{5} - \varepsilon\right) \sum_{\mathfrak{M}-1 < n \leq x} \frac{1}{n \log n}. \end{aligned}$$

Thus (without loss of generality we may assume that  $x$  is an integer) for  $x \geq \mathfrak{M}$ , we have

$$\begin{aligned} \mathfrak{S}(x) &\leq \mathfrak{S}(\mathfrak{M} - 1) + \Sigma_1(x) - \Sigma_1(\mathfrak{M} - 1) - \Sigma_0(x) + \Sigma_0(\mathfrak{M} - 1) \\ &\quad - \left(\frac{4}{5} - \varepsilon\right) \left(\Sigma_{-1}(x) - \Sigma_{-1}(\mathfrak{M} - 1)\right). \end{aligned}$$

Now, considering above lemmas and the restriction  $0 < \varepsilon < 4/5$ , for  $x \geq \mathfrak{M}$ , we obtain

$$\mathfrak{S}(x) \leq \frac{1}{2} \log^2 x - \log x - \left(\frac{4}{5} - \varepsilon\right) \log \log x + C_u + R_u,$$

where

$$C_u = \mathfrak{S}(\mathfrak{M}-1) - \Sigma_1(\mathfrak{M}-1) + \Sigma_0(\mathfrak{M}-1) + \left(\frac{4}{5} - \varepsilon\right) \Sigma_{-1}(\mathfrak{M}-1) + C_1 - C_0 - \left(\frac{4}{5} - \varepsilon\right) C_{-1},$$

and

$$R_u = \frac{6x \log x + 13 \log x + 25}{12x^2} + \frac{3x + 1}{6x^2} + \left(\frac{4}{5} - \varepsilon\right) \frac{3x \log x + \log x + 1}{6x^2 \log^2 x}.$$

Now, we put  $\varepsilon = 1/12423$ , which gives  $\mathfrak{M} = 32299$ . Then  $C_u = 6.488573380484\dots$ , and  $R_u$  is a decreasing function of  $x$  for  $x \geq \mathfrak{M}$ , which yields  $R_u \leq 0.000177415261\dots$ . Therefore, we obtain

$$\mathfrak{S}(x) \leq \frac{1}{2} \log^2 x - \log x - 0.799919504145 \log \log x + 6.488750795746 \quad (x \geq 32299),$$

which holds true for  $2 \leq x \leq 32298$  by Maple computations.

Similarly, we find the lower bound; Corollary 2.4 of HASSANI [1], asserts that for every  $x \geq 7$ , we have

$$\pi(x) \leq \frac{x}{\log x - 1 - \frac{151}{100 \log x}}.$$

This result and the above lemmas yield for every  $x \geq 7$  that

$$\mathfrak{S}(x) = \mathfrak{S}(6) + \mathfrak{S}(6, x) \geq \mathfrak{S}(6) + \sum_{6 < n \leq x} \frac{\log n}{n} - \sum_{6 < n \leq x} \frac{1}{n} - \frac{151}{100} \sum_{6 < n \leq x} \frac{1}{n \log n},$$

and our Lemmas give  $\mathfrak{S}(x) \geq \frac{1}{2} \log^2 x - \log x - 1.51 \log \log x + C_\ell + R_\ell$  for every  $x \geq 7$ , with

$$C_\ell = \mathfrak{S}(6) - \Sigma_1(6) + \Sigma_0(6) + 1.51 \Sigma_{-1}(6) + C_1 - C_0 - 1.51 C_{-1} = 3.734603144333\dots,$$

and

$$R_\ell = -\frac{6x \log x + 13 \log x + 25}{12x^2} - \frac{3x + 1}{6x^2} - 1.51 \frac{3x \log x + \log x + 1}{6x^2 \log^2 x} \geq -0.358785749003\dots,$$

where the last inequality holds for  $x \geq 7$ . Therefore, we obtain

$$\mathfrak{S}(x) \geq \frac{1}{2} \log^2 x - \log x - 1.51 \log \log x + 3.375817395331 \quad (x \geq 7),$$

which holds true for  $x = 6$ , too. Also, reducing the value of constant, Maple computations verifies the following

$$\mathfrak{S}(x) \geq \frac{1}{2} \log^2 x - \log x - 1.51 \log \log x + 0.8994 \quad (x \geq 2).$$

This completes the proof.

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