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PERFECTLY ORDERABLE GRAPHS AND UNIQUE COLORABILITY

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Given a linear order $<$ on the vertices of a graph, an *obstruction* is an induced P_4 $abcd$ such that $a < b$ and $d < c$. A linear order without any obstruction is called *perfect*. A graph is *perfectly orderable* if its vertex set has some perfect order. In the graph G, for two vertices x and y, x *clique-dominates* y if every maximum size clique containing y, contains x too. We prove the following result: *If a perfectly orderable graph is clique-pair-free then it contains two vertices such that one of them clique-dominates the other one*

1. PRELIMINARIES AND INTRODUCTION

The graphs here will be simple and undirected. We denote a graph with vertex set V and edge set E by (V, E). A *clique* in a graph is a subgraph with pairwise adjacent vertices. (It need not to be maximal under inclusion.) Let $\omega(G)$ denote the maximum size of a clique in G. A clique of size k will be called a k-clique. A *stable set* (independent set) is a vertex set with pairwise nonadjacent vertices.

A **clique-pair** in a graph G with $\omega(G) = \omega$: two cliques of size ω, whose intersection has size $\omega - 1$. A graph is **clique-pair-free** if it has no clique-pair. The pair of vertices xy is ω*-critical* in G if $\omega(G + xy) > \omega(G)$. Clearly, this is equivalent to the following property:

i) there exists a clique-pair (X, Y) such that $X - Y = \{x\}, Y - X = \{y\}$.

Obviously, a graph is clique-pair-free if and only if it has no ω-critical pair. The α-critical edges are the complementary objects. They are much more known in the literature [**13**] but in this work we need rather ω-critical pairs.

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A (proper) (vertex-) coloration of a graph is a partition of the vertex set into stable sets. Let $\chi(G)$ be the minimum number of colors in a coloration of G (the *chromatic number* of G). A graph G is **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph H of G. A graph is **uniquely colorable** if its vertex set has only one partition which is a minimum coloration.

For a graph G, its *clique rank* $r(G)$ is the linear rank of the set

$\{v_Q : v_Q$ is the characteristic vector of a maximum-size clique Q in $G\}$.

A *minimal imperfect graph* is a graph which is not perfect and, deleting any vertex from it, the remaining graph is perfect. The **Strong Perfect Graph Theorem** states that a minimal imperfect graph is an odd induced cycle on at least 5 vertices or its complement.

Given a linear order $<$ on the vertices of a graph, an *obstruction* is an induced P_4 $abcd$ such that $a < b$ and $d < c$. A linear order without any obstruction is called *perfect*. A graph is *perfectly orderable* if its vertex set has some perfect order. (We do not use the same terminology as [**6**] but the definition of perfectly orderable graphs here is equivalent to the definition there, due to *Theorem* A.)

Greedy (*First Fit*) *Coloring* of an ordered graph:

Colors: 1,2, ...

First vertex: color 1.

We color the vertices in the order which is given.

Color of a new vertex x: minimal color, not given to a former neighbor of x.

In the graph G, for two vertices x and y, x *clique-dominates* y if every maximum size clique containing y, contains x too.

Remark It is allowed that the two points are contained in the same ω-cliques and also that y is not contained in any ω-clique.

Definition. *A graph G is partitionable if for every vertex v of G, there exists a coloring of the graph $G-v$ with color classes of the same size and the complementary graph $\overline{G-v}$ has the same property.*

2. INTRODUCTION TO UNIQUELY COLORABLE GRAPHS AND CLIQUE-PAIRS

After more than forty years, the Strong Perfect Graph Conjecture (SPGC) has been proved [**5**]. (Thus, it can be called Theorem (SPGT).) Several new branches of combinatorics were inspired or even generated by the SPGC.

Here we formulate three strong results, related to the SPGT. The definitions for the Introduction can be found in Section 1.

Perfect Graph Theorem (Lovász, 1971, [**10**]). *A graph is perfect if and only if its complement is perfect.*

Lovász's Theorem (1972, [**11**]). *If a graph is minimal imperfect, then it is partitionable.*

We have chosen the following form of PADBERG's theorem because it is the most convenient form for our purposes:

Theorem (PADBERG, 1974, [**14**]). *If we delete an arbitrary vertex of a minimal imperfect graph, then the remaining graph is uniquely colorable.*

We mention two implications:

SPGT⇒LOVÁSZ's Theorem⇒Perfect Graph Theorem

PADBERG's theorem suggests: a possible way to utilize the fact that deleting any vertex, we obtain a perfect graph, is

to look for more information about perfect uniquely colorable graphs.

This direction of research has been started in a paper of TUCKER [**17**]. The papers [**9**], [**8**], [**16**] and [**12**] followed this line. Among the possible benefits of these investigations, we can mention here the hope of nice combinatorial methods for coloring perfect graphs.

Many specific conjectures were published (or announced) in this subject. There are conjectures on the combinatorial characterization of uniquely colorable perfect graphs, as well. Some of them have been refuted, e.g., for the so-called Bold Conjecture, SAKUMA [**15**] has given a counterexample.

A simple linear algebraic characterization exists already [**8**]:

Rank Theorem. *A perfect graph G is uniquely colorable if and only if* $r(G) = |V(G)| - \omega(G) + 1$.

Clique-Pair-Conjecture (**CPC** [**8**]). *If G is a uniquely colorable perfect graph, and not a clique, then it contains a clique-pair.*

This conjecture is weaker than all the others and is still not proved. (Here we note that the condition of perfectness cannot be omitted.)

For comparability graphs, the so-called MEYNIEL-graphs, claw-free perfect graphs and the complements of all these graphs, the **CPC** is proved [**17**], [**2**], [**1**].

The deepest result in connection with the **CPC** is

Theorem [**9**]. *The CPC is true for graphs G with $\omega(G) \leq 3$.*

The class of perfectly orderable graphs was introduced by CHVÁTAL [**6**]. (The definition see above.) It is a large class, containing all the comparability graphs, the triangulated and co-triangulated graphs. From another point of view, these graphs have nice properties, e.g., the whole class is contained in BIP* and in the class of strongly perfect and strongly quasi-parity graphs as well. (See [**7**], [**4**]). The most important property of perfectly orderable graphs is the following result of CHVÁTAL [**6**]:

Theorem A. $(G, <)$ *is a perfect order* $\Leftrightarrow$ *for every induced ordered subgraph H of G, the Greedy Coloring, applied to $(H, <)$, gives a minimum coloration of H with $\omega(H)$ colors.*

In this note unfortunately we cannot give the proof of the **CPC** for perfectly orderable graphs, only a partial result:

Theorem. *If a perfectly orderable graph is clique-pair-free then it contains two vertices such that one of them clique-dominates the other one.*

3. PROOF OF THE THEOREM

First we formulate a Lemma proved and used by Chvátal in [**6**].

Lemma 1. *Let* $G = (V, E)$, $(G, <)$ *be a perfect order*, R *be a clique*, S *be a stable set in* G, *disjoint from* R. *Furthermore*, *let the following condition be satisfied*:

For every $r \in R$, *there exists an* $s \in S$ *with* $rs \in E$ *and* $s < r$. **(*)**

Then we have an $s_0 \in S$ *such that* $s_0 r \in E$ *for every* $r \in R$.

We prove Theorem 1 by way of contradiction: we suppose that there is no clique-dominating pair in our graph. Let C be the set of those vertices which get color 1 in the Greedy Coloring on some perfect order $<$ of G and let x be the vertex colored last in C. If there exists no ω-clique containing x then we are done since any other vertex clique-dominates x, by definition. Else, let Q be an ω-clique, containing x.

First case. Condition **(*)** of Lemma 1 holds for $R = Q - x$ and $S = C - x$. In this case there exists a vertex c_0 in S such that $c_0 r \in E$ for every $r \in R$. Then $c_0 \neq x$ and, obviously, $Q \cup c_0$ induces a clique-pair (in other words, c_0 and x yield an ω-critical pair), a contradiction.

Second case. Condition **(*)** of Lemma 1 does not hold for $R = Q - x$ and $S = C - x$ that is, there is a vertex y in R for which no s in S such that $ys \in E$, $s < y$. The color of y is not color 1, so the Greedy Coloring was not able to color it by color 1, that means, there was a neighbor $c < y$ of y, colored already by color 1. If $c \neq x$ then we have a contradiction – Condition **(*)** is satisfied by the vertex y. We may suppose $c = x$. Thus $y > x$. If x clique-dominates y then we are done. Otherwise, we have some ω-clique K with $y \in K$, $x \notin K$. By Theorem A, $\chi(G) = \omega(G)$ and K contains a vertex c_1 colored by color 1. If $c_1 < y$ then y satisfies Condition (*), though we are in the Second case, consequently $c_1 > y$. But then $c_1 > x$, contradicting the definition of x. The proof of the theorem is done. □

Remark to the references. Some papers can be found in Topics on Perfect Graphs [**3**], though appeared originally elsewhere. We refer to this volume in such cases.

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