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ON TYPES, FORM AND SUPREMUM OF THE SOLUTIONS OF THE LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER WITH ENTIRE COEFFICIENTS

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Some new, basically combined classical procedures for qualitative analysis of the equation

$$y'' + a(x)\,y' + b(x)\,y = 0,$$

if $a(x)$ and $b(x)$ are continuously differentiable coefficients, are given in this paper, in the sense of general form of the solution, integral equations for the forms of the solution and estimation of the supremum of the solution.

The aim of this paper is to show that the variety of solutions of the very important differential equation

$$y'' + a(x)\,y' + b(x)\,y = 0, \tag{1}$$

if the coefficients are entire on the semi-infinite line $[0, +\infty)$, is very small and that the solutions are actually divided into two distinct classes: the first class include monotonous solutions of exponential type, the second class comprise oscillating functions, taken into account generally, very similar to ordinary $\sin x$ and $\cos x$.

The huge variety of the solutions of equation (1), well-known from the Theory of Special Functions, is in most cases a consequence of singularities of the coefficients $a(x)$ and $b(x)$ as well as the leading coefficient, $A(x)$, in the formula (1.A) at the end of the paper.

The famous thought of the great analyst Gurs: "Give me singularities and I will tell you about the function" is confirmed by this.

There are two basic classes of solutions of equation (1) with entire coefficients:

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I. Monotonous solutions in the form of generalized hyperbolic functions

$$\cosh_{(a,b)} x \text{ and } \sinh_{(a,b)} x$$

II. Oscillating solutions in the form of generalized trigonometric functions

$$\sin_{(a,b)} x \text{ and } \cos_{(a,b)} x$$

By the iterative method basic existential and numerical approaches are given by means of sums - iterations.

We specially emphasize integral equations for equation (1), with possibility for a priori estimation of the supremum of the solution.

It is known ([**4**], [**6**], see also [**3**], [**5**], [**7**]) that the equation (1) is, by the substitution of the function

$$y = e^{-\frac{1}{2}\int a(x)\,\mathrm{d}x} \cdot z, \tag{2}$$

transformed to the canonical form of a second order equation (3),

$$z'' = f(x)\, z, \tag{3}$$

which has only a single coefficient, $f(x)$, which depends on $a(x)$ and $b(x)$ and is given by:

$$f(x) = \frac{a^2(x)}{4} + \frac{a'(x)}{2} - b(x)\,. \tag{4}$$

Furthermore it is also known [**3**] that the substitution

$$z' = u \cdot z \tag{5}$$

transforms (3) to the Riccati nonlinear equation of the first order,

$$u' + u^2 = f(x)\,. \tag{6}$$

The latter form is also canonical and includes the same coefficient $f(x)$ with no change [3]. From (6) there is

$$u^2 = f(x) - u'$$

with the presumption $f(x) > 0$. (If $f(x) < 0$, the procedure is similar, but there is a need to consider $u' < 0$ in that case).

In order to have real solutions of equation (6) in terms of u the basic inequality $f(x) - u' \geq 0$ must be fulfilled as well. This implies

$$u'(x) \leq f(x)\,. \tag{7}$$

This is the basis tool for the estimation of type and supremum of all the equations, by means of the method we have not yet seen in the literature.

Since equation (1) has two arbitrary coefficients, $a(x)$ and $b(x)$, we make the following presumption for the coefficients: $a(x)$ and $b(x)$ are continuously differentiable on $[0,+\infty)$. Since the solutions theoretically include two arbitrary integration constants, C_1 and C_2, which may not remain undetermined with estimations – because of the estimations are irrelevant at case – the best way to approach the problem is to divide the procedure into several phases starting with some basic but important results:

The first special case: the equality in (7)

It is shown that this case, although known from before, is very important for further estimations.

In (6) let $u' = f(x)$; from (6) this implies that $u^2 = 0$, i.e. the equality

$$u = 0 \tag{8}$$

From (5) $u = \dfrac{z'}{z}$ it follows that $z' = 0$ and from (2) we get the solution of equation (1)

$$y = C\exp\Big(-\frac{1}{2}\textstyle\int a(x)\,\mathrm{d}x\Big).$$

Equation (6) also implies that $f(x) = 0$, i.e. according to (4) there is a relation between the coefficients. So that $b(x)$ depends on $a(x)$ as

$$b(x) = \frac{a^2(x)}{4} + \frac{a'(x)}{2}. \tag{9}$$

So from (3) it follows that $z(x)$ is trivial, being defined from

$$z'' = 0, \text{ i.e. } z = C_1 x + C_2, \tag{10}$$

and from (2) a general solution of the equation (1) is obtained

$$y(x) = (C_1 x + C_2)\, e^{-\frac{1}{2}\int a(x)\,\mathrm{d}x}, \tag{11}$$

a common yet important boundary result. The constants C_1 and C_2 are determined from the initial conditions: $y(0)$ and $y'(0)$. Thus we have

$$C_2 = y(0) \text{ and } C_1 = y'(0) + \frac{a(0)}{2} y(0)$$

and the solution is

$$y(x) = \Big(\Big(y_0' + \frac{a_0}{2} y_0\Big) x + y_0\Big)\, e^{-\frac{1}{2}\int a(x)\,\mathrm{d}x}. \tag{12}$$

From the latter a simple estimation for the solutions in the interval $[0,x]$ implies:

$$y(x) < \Big(\Big|y_0' + \frac{a_0}{2} y_0\Big|\, x + |y_0|\Big)\, e^{-\frac{1}{2}\int \min a(x)\,\mathrm{d}x}.$$

The conclusion is that the sign of the coefficient $a(x)$ is of crucial importance for the behavior of the solution $y(x)$.

1° If $a(x) < 0$, then

$$-\frac{1}{2}\int_0^x a(x)\,\mathrm{d}x\mathrm{d}y < \frac{1}{2}\max|a(x)|$$

and the estimation is

$$y(x) < |C_1x + C_2|\,e^{\frac{x}{2}\max a(x)}, x \in [0, x]$$

2° if $a(x) > 0$ the solutions is bounded, and

$$y(x) < |C_1x + C_2|\,e^{-\frac{x}{2}\min a(x)}$$

i.e., there is also an asymptotic behavior of the solution: $y(x) \to 0, x \to \infty$.

We obtained a trivial but bounding and determining result:

Theorem 1. *Differential equation* (1), *in which* $b(x)$ *depends on* $a(x)$ *as in* (9), *namely*

$$y'' + a(x)\,y' + \left(\frac{a^2(x)}{4} + \frac{a'(x)}{2}\right)y = 0,$$

has the general solution, (11), *and in respect to initial conditions*, $y(0)$ *and* $y'(0)$, *the form of the solution is* (12); *the estimations and asymptotic behaviors given in* $(12, 1°)$ *and* $(12, 2°)$ *are possible.*

The second special case

Let $b(x)$ depend on $a(x)$ and the derivative, $a'(x)$, but not explicitly:

$$b(x) = \frac{a^2(x)}{4}. \tag{13}$$

Then there is the special equation (1) with only one variable function $a(x)$,

$$y'' + a(x)\,y' + \frac{a^2(x)}{4}\,y = 0, \tag{14}$$

in which case the coefficient in the equation (3) is $f(x) = \dfrac{a'(x)}{2}$ and (3) transforms to

$$z'' = \frac{a'(x)}{2}\,z. \tag{15}$$

The first general case

If $a(x) > 0$, i.e., $a(x)$ increases monotonically, then it is known that (15) determines classes of exponential solutions for $z(x)$ and (according to (2)) $y(x)$ (see [**2**]). Two integrations in (15) give the integral form

$$z(x) = C_1 x + C_2 + \frac{1}{2}\int_0^x\int_0^x a'(x)\, z(x)\, \mathrm{d}x^2 \tag{16}$$

from which, through iterations, we obtain

$$\begin{aligned} z(x) &= C_1\left(x + \frac{1}{2}\int_0^x\int_0^x xa'(x)\,\mathrm{d}x^2 + \frac{1}{2^2}\int_0^x\int_0^x a'(x)\int_0^x\int_0^x xa'(x)\,\mathrm{d}x^4 + \cdots\right) \\ &\quad + C_2\left(x + \frac{1}{2}\int_0^x\int_0^x a'(x)\,\mathrm{d}x^2 + \frac{1}{2^2}\int_0^x\int_0^x a'(x)\int_0^x\int_0^x xa'(x)\,\mathrm{d}x^4 + \cdots\right), \end{aligned}$$

where, because of $a'(x) > 0$, the members of the sum of iterations in the form of multiple integrals monotonically increase and the sums above determine nonelementary hyperbolic functions with basis $\sqrt{\dfrac{a'(x)}{2}}$. The sums are symbolically denoted as

$$z(x) = C_1 \sinh_{\sqrt{a'(x)/2}} x + C_2 \cosh_{\sqrt{a'(x)/2}} x.$$

The corresponding solution of the equation (1) is

$$y(x) = e^{-\frac{1}{2}\int a(x)\,dx}\left(C_1 \sinh_{\sqrt{a'(x)/2}} x + C_2 \cosh_{\sqrt{a'(x)/2}} x\right). \tag{17}$$

Theorem 2. *Equation* (14), *where* $a(x)$ *is a continuously differentiable coefficient with the feature* $a'(x) > 0$, *has a solution in the form of fast rising sum defining hyperbolic functions* (17), *in which for a finite* x *a upper bound could be determined in* $[0, x]$, *dependent on minima or maxima of the functions* $a(x)$ *and* $a'(x)$; *it has the feature*

$$y(x) \to \infty,\ x \to \infty.$$

The second general case

Much more important, yet more difficult to analyze, is the case when in (15) $a'(x) < 0$, i.e., if we put

$$\frac{a'(x)}{2} = -F(x), \tag{18}$$

where $F(x) > 0$ and recall the well-known theorem on existence of oscillating solutions which states: If in the equation $z'' + F(x)z = 0$

$$1^\circ\ F(x) > 0 \text{ and } 2^\circ \int_0^{+\infty} F(x)\,\mathrm{d}x \text{ diverges,}$$

then all of its solutions are oscillating.

In our case it is:

1° $F(x) = -\dfrac{a'(x)}{2} > 0$ by presumption, and

2° $\int\limits_0^{+\infty} F(x)\,\mathrm{d}x = -\int\limits_0^{+\infty} \dfrac{a'(x)}{2}\,\mathrm{d}x = -\dfrac{1}{2}\,a(x)|_0^{+\infty} = \dfrac{1}{2}\big(a(0) - a(\infty)\big) = \infty.$

Thus the conditions are met and the equation has only oscillating solutions. If the equation is written in the normal form

$$z'' = -F(x)\,,\ F(x) > 0,$$

then in the form of an integral equation

$$z(x) = C_1 x + C_2 - \int\limits_0^x\int\limits_0^x F(x)\,z(x)\,\mathrm{d}x^2$$

in which the iterations alter sign consecutively, writing

$$\begin{aligned} z(x) \ = C_1\Bigg(& x - \int\limits_0^x\int\limits_0^x xF(x)\,\mathrm{d}x^2 + \int\limits_0^x\int\limits_0^x F(x)\int\limits_0^x\int\limits_0^x xF(x)\,\mathrm{d}x^4 \\ & - \int\limits_0^x\int\limits_0^x F(x)\int\limits_0^x\int\limits_0^x F(x)\int\limits_0^x\int\limits_0^x xF(x)\,\mathrm{d}x^6 + \cdots\Bigg) \\ + C_2\Bigg(& 1 - \int\limits_0^x\int\limits_0^x F(x)\,\mathrm{d}x^2 + \int\limits_0^x\int\limits_0^x F(x)\int\limits_0^x\int\limits_0^x F(x)\,\mathrm{d}x^4 \\ & - \int\limits_0^x\int\limits_0^x F(x)\int\limits_0^x\int\limits_0^x F(x)\int\limits_0^x\int\limits_0^x F(x)\,\mathrm{d}x^6 + \cdots\Bigg). \end{aligned}$$

It has been proved [**1**] that the iterations determine a second – order trigonometry by means of basis generalized trigonometric functions, called generalized sine with base $F(x)$ and generalized cosine with base $F(x)$.

$$\begin{aligned} \cosh_{F(x)} x = 1 & - \int\limits_0^x\int\limits_0^x F(x)\,\mathrm{d}x^2 + \int\limits_0^x\int\limits_0^x F(x)\int\limits_0^x\int\limits_0^x F(x)\,\mathrm{d}x^4 \\ & - \int\limits_0^x\int\limits_0^x F(x)\int\limits_0^x\int\limits_0^x F(x)\int\limits_0^x\int\limits_0^x F(x)\,\mathrm{d}x^6 + \cdots \end{aligned} \tag{21}$$

$$\begin{aligned} \sinh_{F(x)} x = x & - \int\limits_0^x\int\limits_0^x xF(x)\,\mathrm{d}x^2 + \int\limits_0^x\int\limits_0^x F(x)\int\limits_0^x\int\limits_0^x xF(x)\,\mathrm{d}x^4 \\ & - \int\limits_0^x\int\limits_0^x F(x)\int\limits_0^x\int\limits_0^x F(x)\int\limits_0^x\int\limits_0^x xF(x)\,\mathrm{d}x^6 \cdots\,, \end{aligned} \tag{22}$$

i.e., there is the solution

$$z(x) = C_1 \sinh_{F(x)} x + C_2 \cosh_{F(x)} x. \tag{21}$$

Consequently the general solution to the equation (1) is

$$y(x) = e^{-\frac{1}{2}\int a(x)\,\mathrm{d}x}\big(C_1 \sinh_{F(x)} x + C_2 \cosh_{F(x)} x\big). \tag{22}$$

Theorem 3. *The equation* (13), *in which* $a(x)$ *has the feature* $a'(x) < 0$ (*i.e.* $a(x)$ *monotonically decreases*), *has oscillating solutions in the form* (22).

Since the similarity of the generalized trigonometric functions to the ordinary sine and cosine could be shown by means of the iteration method in the sums (19) and (20), with the argument $x\sqrt{F(x)}$ instead, i.e., the following relations can be derived

$$\cos_{F(x)} x \approx \cos\left(x\sqrt{F(x)}\right) \text{ and } \sin_{F(x)} x \approx \sin\left(x\sqrt{F(x)}\right), \tag{23}$$

the function (21) is bounded and the integral (22), in which $a(x)$ monotonously decreases, converges to zero.

Theorem 4. *Equation* (14), *when* $x \to \infty$, *allows the following asymptotic behaviour of the solutions*:

- *if* $a'(x) > 0$, *then every solution is unbounded and monotonic,*
- *if* $a'(x) < 0$, *then every solution converges to zero and is oscillating,*
- *if* $a'(x) = 0$, *i.e.* $a(x) =$ *Const, then the solutions are* $y = e^{-C/2}(C_1 x + C)$,

and there are both monotonic and nonmonotonic, as well as both bounded and unbounded solutions. (*This is the consequence of the fact that characteristic equation for* $y'' + Cy' + \dfrac{C^2}{4}y = 0$ *reads* $r^2 + Cr + \dfrac{C^2}{4}r = 0$, *and has a double root* $r = -\dfrac{C}{2}$. *Thus the asymptotic is dependent on the sign of the constant* C).

The third case. Estimation of the solutions

Let there now be no special restrictions in the form of rigid relations between the coefficients $a(x)$ and $b(x)$.

Then, beside (1), the following is valid as well $u^2 = f(x) - u' > 0$ (the case $u^2 = 0$ has been analyzed in the first case). Therefore, let there be the strict inequality $u' < f(x)$.

Since the solving of differential inequalities by means of integrating the left and the right side makes a lot of difficulties, beside the determination of the constants. For the latter differential inequality we substitute the differential equation

$$u' = f(x) - \varepsilon(x), \tag{24}$$

where $\varepsilon(x)$ is an unknown function (also dependent on $u(x)$), yet for which is known that: 1° $\varepsilon > 0$ (i.e. $\varepsilon = u^2$); 2° $\varepsilon(x)$ is a supplement for $u'(x)$, for it to reach $f(x)$. From (24) implies the integration

$$u = \int_0^x \big(f(x) - \varepsilon(x)\big)\,\mathrm{d}x + C_1,$$

where $C_1 = u(0) = \text{Const}$. Since (4) is valid for $f(x)$, hence

$$u = \int\limits_0^x \left(\frac{a^2}{4} + \frac{a'}{2} - b - \varepsilon(x)\right) \mathrm{d}x + u_0$$

and, according to (5), we get a new differential equation with regard to the function z

$$\frac{z'}{z} = \int\limits_0^x \left(\frac{a^2}{4} + \frac{a'}{2} - b - \varepsilon(x)\right) \mathrm{d}x + u_0.$$

After integration we get

$$\ln z = \int\limits_0^x \int\limits_0^x \left(\frac{a^2}{4} + \frac{a'}{2} - b - \varepsilon(x)\right) \mathrm{d}x^2 + u_0 x + C_2 \tag{25}$$

Since from (2) it follows that $z = y\, e^{\frac{1}{2}\int a(x)\,\mathrm{d}x}$, we get

$$y \cdot e^{\frac{1}{2}\int a(x)\,\mathrm{d}x} = e^{u_0 x + C_2} \cdot e^{\int\limits_0^x \int\limits_0^x (f(x) - \varepsilon(x))\,\mathrm{d}x^2}$$

where C_2 is obtained from (25) substituting $x = 0$ and $\ln z_0 = C_2$; with regard to (2) we have $C_2 = \ln y(0)$. Now we get

$$y = e^{-\frac{1}{2}\int a(x)\,\mathrm{d}x} \cdot e^{u_0 x + \ln y_0 + \int\limits_0^x \int\limits_0^x (f(x) - \varepsilon(x))\,\mathrm{d}x^2},$$

where, if we replace $f(x)$ with its value (4), it remains

$$y(x) = y_0 \cdot e^{u_0 x + \int\limits_0^x \int\limits_0^x \left(\frac{a^2(x)}{4} - b(x) - \varepsilon(x)\right)\mathrm{d}x^2}. \tag{26}$$

Since $u = \dfrac{z'}{z}$ and $z' = y' e^{\frac{1}{2}\int a(x)\,\mathrm{d}x} + \dfrac{a(x)}{2}\, y e^{\frac{1}{2}\int a(x)\,\mathrm{d}x}$,

$$u = \frac{y'}{y} + \frac{a(x)}{2} \text{ and } u_0 = \frac{y_0'}{y_0} + \frac{a_0}{2}.$$

For the solution of (1) we get integral expression

$$y(x) = y_0 \cdot e^{\left(\frac{y_0'}{y_0} + \frac{a_0}{2}\right) + \int\limits_0^x \int\limits_0^x \left(\frac{a^2(x)}{4} - b(x) - \varepsilon(x)\right)\mathrm{d}x^2}, \tag{27}$$

where the solution of (1) is given in the form (27) with one undetermined element, and in the function of all other determined and known elements

$$a(x),\ b(x)\ y_0,\ y_0'. \tag{28}$$

The solution (27) is suitable for a qualitative analysis. This way, the following is formulated:

Theorem 5. *Every solution of the linear homogeneous differential equation of the second order* (1) *with continuously differentiable coefficients* $a(x)$ *and* $b(x)$ *could be expressed through exponential integral form* (27).

If we denote K a constant composed of given and known values $a(0), y(0), y'(0)$: $K = \dfrac{y_0'}{y_0} + \dfrac{a(0)}{2}$ and always $\dfrac{a^2}{4} > 0$ and $\varepsilon > 0$, the following estimations of the general solution (27) are obtained

$$y_0 \cdot e^{kx - \int\limits_0^x \int\limits_0^x \left(b(x) + \varepsilon^2(x)\right) \mathrm{d}x^2} \le y(x) \le y_0 \cdot e^{kx + \int\limits_0^x \int\limits_0^x \left(\frac{a^2(x)}{4} - b(x)\right) \mathrm{d}x^2}, \tag{29}$$

where only the right side is ultimately determined (in the same time it must be that $f(x) > 0$, i.e. $\dfrac{a^2}{4} - b > 0$).

Equation (27) (because ε is variable) is actually a nonlinear integral equation derived from the linear equation (1) and corresponding RICCATI equation (6) under condition (4). Inequalities better than (27) could be obtained if we notice in (27) – which is a correct equality – that

$$\varepsilon(x) = u^2 = \left(\frac{y'}{y} + \frac{a(x)}{2}\right)^2.$$

Therefore, we have

$$y(x) = y_0 \cdot e^{kx - \int\limits_0^x \int\limits_0^x \left(b(x) + a(x)\frac{y'}{y} + \left(\frac{y'}{y}\right)^2\right) \mathrm{d}x^2}. \tag{30}$$

If we analyze functional of a quadratic function

$$P_2(y, y') = P_2\left(a, b, \frac{y'}{y}\right) = P_2(Y) = b + aY + Y^2;\ Y = \frac{y'}{y},$$

it is easily found that it has a minimum for $P_2(Y) = 0, a + 2Y = 0, Y = -\dfrac{a}{2}$ and $\min P_2 = P_2\left(-\dfrac{a}{2}\right) = b - \dfrac{a^2}{4}$, for $\dfrac{y'}{y} = -\dfrac{a(x)}{2}$.

However, the integration then gives $y = e^{-\int a(x)\,\mathrm{d}x}$, which is the special solution from the first special case.

This implies that the right side is biggest when we subtract the least, which occurs for $\min P_2(Y)$, where $Y = \dfrac{y'}{y}$, the minimum being equal to $b - \dfrac{a^2}{4}$ for the solution $y = C_1 e^{-\frac{1}{2}\int a(x)\,\mathrm{d}x}$, i.e. for the equation (1) for which (9) is valid as well. This implies:

Theorem 6. *The supremum of all solutions of the equation* (1) *is*

$$y(x) \le y_0 \cdot e^{kx - \int_0^x \int_0^x \left(b(x) - \frac{a^2(x)}{4} \right) \mathrm{d}x^2} . \tag{31}$$

It implies that all the other estimations of the solution $y(x)$ which has no zeroes, derived from the integral equation (30), for all the other possible solutions $y(x) \neq 0$ and for all continuous coefficients $a(x)$ and $b(x)$, are bigger than the above supremum, $y(x)$.

However, since for the minimum of the functional we get a supremum for the solution $u = 0$ i.e., the first special case; i.e., for the pair (a, b) the relation (9) is valid, we have

$$y(x) \le y_0 \cdot e^{kx - \int_0^x \int_0^x \frac{a'(x)}{2} \mathrm{d}x^2}$$

or in a simpler form

$$y(x) \le y_0 \cdot e^{kx - \int_0^x (a(x) - a(0)) \, \mathrm{d}x},$$

or, considering the constant K, we get the least upper limit of all the possible upper limits for equation (1) determined by the pair $(ax), b(x))$:

$$y(x) \le y_0 \cdot e^{\left(\frac{y_0'}{y_0} + a_0 \right) x} \cdot e^{-\frac{1}{2} \int_0^x a(x) \, \mathrm{d}x} . \tag{32}$$

Remark. The fourth case. Possible discontinuous coefficients

Let there be the equation

$$A(x) y'' + a(x) y' + b(x) y = 0, \tag{1.A}$$

equivalent to (1) if

1° $A(x)$ is continuously differentiable in $[0, x]$ and

2° $A(x)$ has no zeroes in $[0, x]$.

Then there is:

$$y'' + \frac{a(x)}{A(x)} y' + \frac{b(x)}{A(x)} y = 0$$

and as an estimation of the solution we have the integral

$$y(x) \le y_0 \cdot e^{kx + \int_0^x \int_0^x \left(\frac{1}{4} \frac{a^2(x)}{A^2(x)} - \frac{b(x)}{A(x)} \right) \mathrm{d}x^2} . \tag{29.A}$$

If $A(x) = 0$ has solutions $x_1, x_2, \ldots$, then all these points must be excluded while iterating and all previous considerations must be repeated in the intervals

$[0, x_1]\,, [x_1, x_2]\,, [x_2, x_3]\,, \ldots$ at which boundary the coefficients $\dfrac{a}{A}$ and $\dfrac{b}{B}$ become undetermined and infinite. The first integral $\int\limits_0^x \left(\dfrac{1}{4}\,\dfrac{a^2(x)}{A^2(x)} - \dfrac{b(x)}{A(x)}\right) \mathrm{d}x$ and the second integral $\int\limits_0^x \int\limits_0^x \left(\dfrac{1}{4}\,\dfrac{a^2(x)}{A^2(x)} - \dfrac{b(x)}{A(x)}\right) \mathrm{d}x^2$ then become improper.

This implies that a huge work remains to be done regarding estimation, existence and convergence of these and improper integrals of this kind; a work that most likely could not be contained within a paper because it actually represents the entire Theory of Special Functions.

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